Parallel computation using generalized models of exactly solvable chaos [1]

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How chaos is useful in the brain information processing is greatly unknown. Here, we show that the statistical property of chaos such as invariant measures naturally organized under a great number of iterations of chaotic mappings can be used for some complex computations, while the precise information of initial conditions which vanishes in the course of iterations deos not matter for this kind of computations. The key observation of the present study is that computation using ergordicity of dynamical systems can be thought of as massively parallel Monte Carlo simulations. Here, to avoid difficulty in elucidating the ergordicity of dynamical systems, we propose computational schemes using the generalized class of one-dimensional chaos with explicit invariant measures. The validity of our results which connect chaos with parallel computation is checked by the precision computations of some transcendental numbers like π.

[1] to be published in RIKEN Review No.14(1996)

One of the most important theoretical challenges in our brain information processing is how a great number of noisy neurons realize the parallelism of computation in the brain. Thus, in relation to this problem, it is of great interest to consider whether or how a kind of chaotic dynamics can be harnessed in brain information processing mechanisms. However, most studies concerning chaos in the brain are based on the empirical computer simulations on some specific conditions. The purpose of the present paper is to show that we can have some ideal model of parallel computation using chaos by purely analytical treatment.

Let us consider the following deterministic process $x_{n+1} = F(x_n)$, where $x_n \in \boldsymbol{M}$. We can say that this dynamical system has the ergordicity if the following relation holds:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} Q(x_i) = \int_{\boldsymbol{M}} Q(x)\mu(dx), \tag{1}$$

where $\mu(dx)$ is the invariant measures on $\boldsymbol{M}$ and $Q(x)$ is a regular function which makes R.H.S. of Eq. (1) finite. Essentially, Monte Carlo simulations can be regarded as evaluating R.H.S. of Eq. (1) by approximately calculating L.H.S. of Eq. (1) under the assumption of the ergordicity. However, it is difficult to get an explicit invariant measure $\mu(dx)$ for a given dynamical system and in general, we cannot tell whether the ergordicity holds or not for the system. Without explicit invariant measures, it is meaningless to consider the Monte Carlo type computations because the integral as R.H.S. of Eq. (1) is unclear.

Quite recently, we discovered the two-parameters class of chaotic dynamical systems with explicit invariant measures[4, 5]. We call this class *exactly solvable chaos* because we can obtain the invariant measures as well as the general solutions analytically. Here, let us briefly review the models of exactly solvable chaos on the unit interval $\boldsymbol{M} = [0, 1] \equiv \boldsymbol{I}$. The first example of exactly solvable chaos was given by Ulam and Von Neumann in 1947 [2] as follows:

$$x_{n+1} = F(x_n) = 4x_n(1 - x_n), \tag{2}$$

where the invariant measure has an explicit expression

$$\mu(dx) = \rho(x)dx = \frac{dx}{\pi\sqrt{x(1-x)}}. \tag{3}$$

Next, in 1985, Katsura and Fukuda[1] generalized the Ulam=Neumann map to the following model:

$$F(x) = \frac{4x(1-x)(1-k^2x)}{(1-k^2x^2)^2} \tag{4}$$

for $0 \leq k < 1$. Its invariant measure can be calculated [4] as

$$\mu(dx) = \rho(x)dx = \frac{dx}{2K(k)\sqrt{x(1-x)(1-k^2x)}}, \tag{5}$$

where $K(k)$ is given by the complete elliptic integral

$$K(k) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{6}$$

The Katsura=Fukuda map for $k = 0$ corresponds to the Ulam=Neumann map. Furthermore, the present author generalized the Katsura=Fukuda to the following model [5]:

$$F(x) = \frac{4x(1-x)(1-lx)(1-mx)}{1-2(l+m+lm)x^2+8lmx^3+(l^2+m^2-2lm-2l^2m-2lm^2+l^2m^2)x^4} \tag{7}$$

with two parameters m and l such that $-\infty < m \leq l < 1$. If we set $m = 0$, we obtain the Katsura=Fukuda map. The explicit invariant measure of the generalized map has the form[4] of

$$\mu(dx) = \rho(x)dx = \frac{dx}{2K(l,m)\sqrt{x(1-x)(1-lx)(1-mx)}}, \tag{8}$$

where $K(l,m)$ is given by the integral

$$K(l,m) = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-lu^2)(1-mu^2)}}. \tag{9}$$

Here, we call (7) *the generalized Ulam=Neumann map*. Furthermore, it is shown [5] that the generalized Ulam=Neumann map (7) is the most generalized model of exactly solvable chaos on $\boldsymbol{I}$, whose Lyapunov exponent is equal to $\log 2$,i.e., we cannot further generalize the map (7). Thus, we can say that the class of the generalized Ulam=Neumann maps is universal in representing exactly solvable chaos on $\boldsymbol{I}$. We note here that their dynamical zeta functions of the generalized Ulam=Neumann maps is the same as the one of the original Ulam=Neumann map[3], by which it means that the statistical characters such as Kolmogorov=Sinai entropy, Lyapunov exponent, and topological entropy do not depend on the parameters l and m. Figure 1 shows the shapes of the generalized Ulam=Neumann maps with different parameters m for $l = 0.5$. Interestingly, their shapes for $m = -20, \cdots, -100$ are very similar to the Poincaré plots of Belousov=Zhabotinski chemical reaction.

Using the explicit expression of the invariant measures $\rho(x)$ of the generalized Ulam=Neumann maps in Eq. (8), we obtain the following formula:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} G(x_n) = \int_0^1 G(x)\rho(x)dx = \frac{1}{2K(l,m)}, \tag{10}$$

where $G(x) = \sqrt{x(1-x)(1-lx)(1-mx)}$. $K(l,m)$ is known to be a transcendental real number like π. Let us consider the problem of parallel computing

of $K(l,m)$ using this model of exactly solvable chaos. We consider P independent processors labeled by $j(1 \leq j \leq P)$. First, we give an initial condition $x_{1,j} \in \boldsymbol{I}$ to each processor. The initial conditions are different each other. Then each processor iterates the map (7) as $x_{n+1,j} = F(x_{n,j})$ to compute the number $\frac{1}{2\hat{K}(x_{1,j})} \equiv \frac{1}{N}\sum_{i=1}^{N} G(x_{n,j})$. Finally, after averaging the above numbers, we have an estimated value of $K(l,m)$ as

$$2\tilde{K}(l,m) \equiv \frac{1}{P}\sum_{j=1}^{P} 2\hat{K}(x_{1,j}) \approx 2K(l,m). \tag{11}$$

To confirm the validity of this proposed parallel algorithm, we computed $2\tilde{K}(l,m)$ for $(l,m) = (0,0), (0.5,-100)$ and $(0.5,-200)$. We set that $P = 100$, $N = 10^6$ and $x_{1,j} = 0.0073333371j$ for $j = 1, \cdots, 100$.

Table 1: Results for $P = 100$ and $N = 1000000$

	$(l,m) = (0,0)$	$(l,m) = (0.5,-100)$	$(l,m) = (0.5,-200)$
$2\tilde{K}(l,m)$	3.14167445904031	0.804839363844501	0.618717074365706
$2K(l,m)$	3.14159265358979	0.804826090564245	0.618817767268870

Table 1 shows that our numerical estimated values $2\tilde{K}(l,m)$ using the standard double precision representations of real numbers neared the exact values $2K(l,m)$ as $|2\tilde{K}(l,m) - 2K(l,m)| < 0.0002$.
Furthermore, as is shown in Fig.2, our estimated values $2\tilde{K}(l,m)$ for $(l,m) = (0.5, 0.3)$ using exactly solvable chaos converge faster than those using the usual Monte Carlo method to evaluate integrals on $\boldsymbol{I}$, although both of the mean square deviations are inversely proportional to the iteration number N. This fastness of convergence of our method over the Monte Carlo method is owing to the fact that $G(x) = \sqrt{x(1-x)(1-lx)(1-mx)} < 1 < \frac{1}{\sqrt{(1-y^2)(1-ly^2)(1-my^2)}}$ for $x, y \in \boldsymbol{I}$ and $0 \leq m \leq l < 1$.

This work was supported by the special researcher's program to promote basic sciences at RIKEN.

References

[1] S. Katsura and W. Fukuda: *Physica* **130 A**, 597(1985).

[2] S. M. Ulam and J. Von Neumann: *Bull. Math. Soc.* **53**, 1120 (1947).

[3] K. Umeno and M. Suzuki: *Phys. Lett.* **A177**, 311(1993).

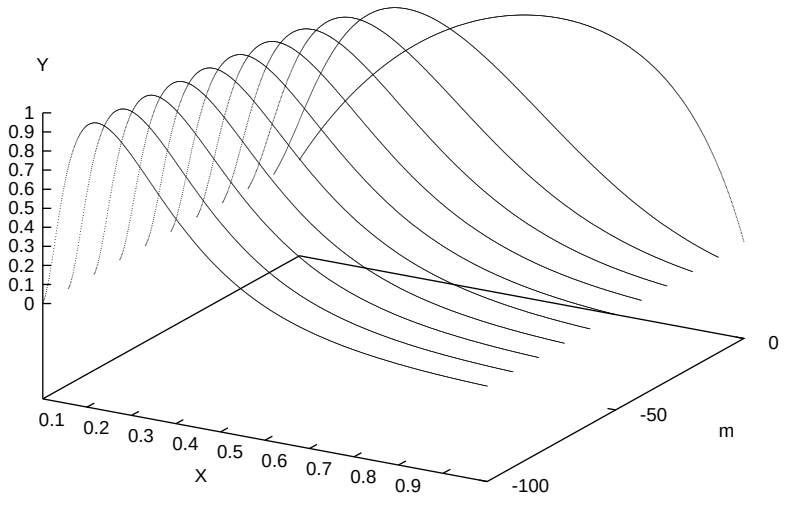
[4] K. Umeno: *Technical Report of IEICE* **NC-96-13**, 23(1996) (In Japanese).

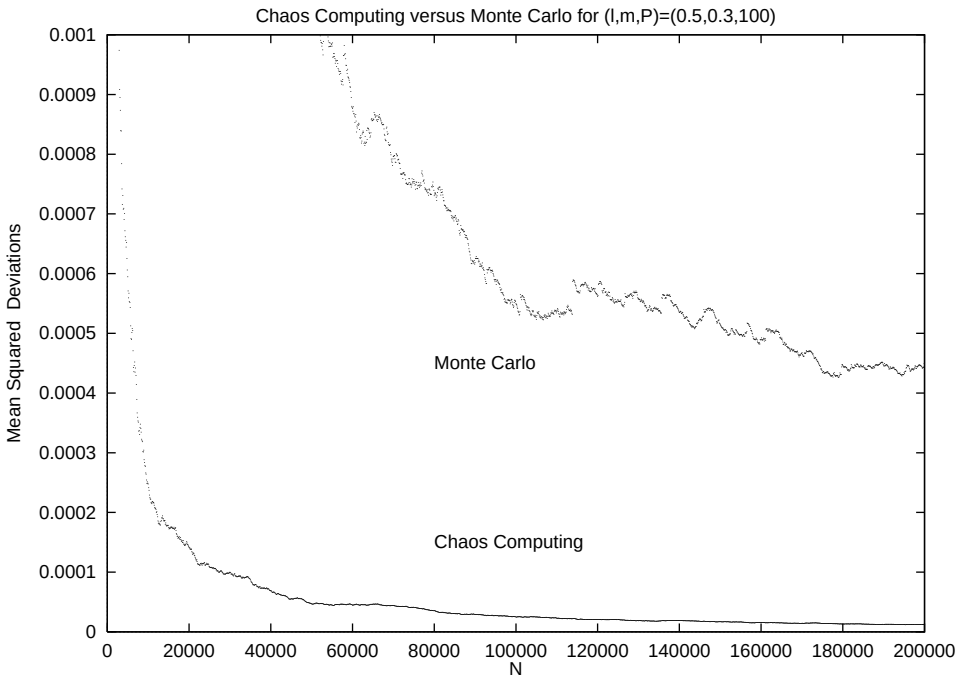
[5] K. Umeno: "The maximal family of exactly solvable chaos", Preprint(1996);

Fig. 1: The generalized Ulam=Neumann maps (7) for $(l, m) = (0.5, 0), (0.5, -10), \cdots, (0.5, -100)$.

Fig. 2: Mean squared deviations defined as $\frac{1}{P}\sum_{j=1}^{P}\{2\hat{K}(x_{1,j}) - 2K(l,m)\}^2$ with $(l,m) = (0.5, 0.3)$ versus the number of iterations N for the present method (Chaos Computing) and for the usual method (Monte Carlo), where the uniform random numbers were generated by the function *ran(seed=564789 11)* in the *FORTRAN77* library to determine the initial conditions $x_{1,j}$ for the Chaos Computing and the Monte Carlo paths.

Generalized Ulam=Neumann Maps for l=0.5




Chaos Computing versus Monte Carlo for (l,m,P)=(0.5,0.3,100)
Mean Squared Deviations
N
Monte Carlo
Chaos Computing
0.001
0.0009
0.0008
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
0
20000
40000
60000
80000
100000
120000
140000
160000
180000
200000